CHAPTER 4

Torus Knots

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ne of our early design quests was to create bracelet patterns that illustrate torus knots. Like ballet dancers who make the difficult look easy, torus knots have a simple elegance that belies a precise choreography. For the bead crochet designer, they are a natural focus, since the various underlying spiral structures in all bracelets form some flavor of torus knot. We begin by explaining what they are and why we find them intriguing, starting with a few preliminary definitions.

Just as every point on a sphere or globe can be identified by its longitude and latitude, every point on a torus can be identified with two coordinates in a similar fashion. On a torus, we measure the two coordinates along longitude and meridian lines, as shown in Figure 4.1. In bead crochet terms, the length of a meridian line is the circumference or thickness of the rope itself, and the length of a longitude line is the circumference or size of the full bracelet. A \([\text{longitude}, \text{meridian}]\) position on a torus also corresponds to an \([\text{X}, \text{Y}]\) position on two perpendicular axes on a flat version of the torus. We will look more closely at the relationship between tori and their flat diagrams later.

A “knot” in this context is a length of string that has been twisted and looped in some way prior to having its two ends tied together (or attached somehow). In a torus knot, the string is looped through and around a torus before having its ends connected so that it can no longer be unlinked from the torus without untying or cutting it first (Figure 4.2).

More precisely, a torus knot is a knot that winds \(P\) times around a torus meridionally and \(Q\) times longitudinally (notated \((P, Q)\)), where \(P\) and \(Q\) are relatively prime. Relatively prime means that \(P\) and \(Q\) are two numbers with no common divisors other than 1, for example, 4 and 7, or 5 and 8. Figure 4.3 shows the torus knots from \(P = 3\) to \(P = 9\) and from \(Q = 2\) to \(Q = 5\). One famous knot you may have seen or heard about is the trefoil, or \((3,2)\) torus knot, shown at the upper left in Figure 4.3. In torus knots, the torus does not have to be present; the string itself is the torus knot, so an imaginary torus is sufficient, as in the \((11, 2)\) torus knot shown in Figure 4.4.

Elements missing from the chart in Figure 4.3 are those where \(P\) and \(Q\) are not relatively prime. You might immediately wonder what’s so special about relatively prime \(P\) and \(Q\): why would we distinguish these cases from instances where \(P\) and \(Q\) have a common divisor? The answer to this question is related to a fun and mysterious fact that has interesting bead crochet design implications, namely, that the “string” in a torus knot can be embedded into or laid onto the torus surface without ever crossing over or under itself, a feat that is physically possible only for a knot with relatively prime \(P\) and \(Q\). In fact, torus knots

\[\text{Longitude line} \quad \text{Meridian line}\]

**FIGURE 4.1** Longitude lines run along the circumference of the full bracelet. Meridian lines run along the circumference of the rope.

* In some texts, the meaning of \(P\) and \(Q\) in this notation is reversed.
are precisely those knots that can be drawn on the surface of a torus without crossings. Furthermore, this ability to “draw” or embed the string on the torus surface without ever crossing itself holds true for any relatively prime $P$ and $Q$, no matter how large! This might seem amazing and mind-bending, unless, of course, you are a mathematician who has spent time playing with such things. Try picking a couple of relatively prime numbers out of a hat, say, $52$ and $27$. Now imagine trying to wrap a single piece of string around a bracelet so that it weaves $52$ times through the hole (i.e., meridionally) and also $27$ times around the full length of the bracelet (i.e., longitudinally) before connecting back to itself at the beginning, all without ever crossing over or under itself. Aside from the problem of avoiding crossings, you might also object that the string had better be thin enough in relationship to the size of the torus—a little tiny torus and a big fat string surely won’t work. However, mathematicians think in theoretical terms and consider the string to be infinitely thin, the width of a single point. But even if we prefer our string to have a tangible width, we can always pick a torus size that is bigger in relationship to the string.

One way to think about a torus knot is to envision a spider traversing the torus and spinning a strand of silk to create the knot as it goes. If the spider travels in precisely the right unvarying direction, namely, at a slope of $Q/P$ on a flat model of the torus, it can produce the desired knot. We shall see diagrams later that illustrate this more clearly, but a slope of $Q/P$ is just the mathematical way of saying that the line changes by $Q$ in the vertical direction as

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<thead>
<tr>
<th>$P$</th>
<th>$P = 3$</th>
<th>$P = 4$</th>
<th>$P = 5$</th>
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<tr>
<td>$Q = 2$</td>
<td><img src="image1.png" alt="Image" /></td>
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<td>$Q = 3$</td>
<td><img src="image7.png" alt="Image" /></td>
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<tr>
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<td>$Q = 5$</td>
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We know that $52$ and $27$ are relatively prime because $27$ is $3^3$ and thus its only prime factor is $3$, which doesn’t divide into $52$.  

**FIGURE 4.3** Examples of torus knots. These torus knot images were produced using the Knotplot software created by Rob Scharein.

**FIGURE 4.4** An $(11, 2)$ torus knot without the torus. This image was produced using the Knotplot software created by Rob Scharein.
it changes by P in the horizontal direction. This is exactly the direction needed to allow the spider to accomplish its goal of precisely Q longitudinal traversals and, simultaneously, P meridian traversals before returning to its starting position, where it can then glue the two ends of the strand together. The spider’s special torus knot choreography is to spin (in both senses) around the torus the requisite number of times, both meridionally and longitudinally, without ever stepping on the same spot twice. While the spider’s task might sound tricky, it is really just following a straight line with a precisely chosen slope, and if you’ve ever made a standard spiral pattern in bead crochet, you may already have created some interesting torus knots.

**Challenge** Can you design a trefoil knot—or (3,2) torus knot—in bead crochet? Can you come up with a general method for designing any (P,Q) torus knot in bead crochet? As in a proper torus knot, the “string” component of the design (in this case some “linear” pattern of contrasting color beads) must never cross itself.

Ideally, the gaps between the “string” should remain symmetric throughout, just as they are in the torus knots shown in Figure 4.3. However, unlike the torus knots in Figure 4.3, the lines need not be perfectly smooth. In fact, given the bead-sized “pixels” on our bracelet “canvas,” some zigzagging is likely needed.

**Construction Using Physical Twists**

If you have a bagel or ceramic donut and simply want to draw a torus knot on it with a marker, it might take some thought to figure out how to do so, even for relatively small values of P and Q. If you have a laundry marker and a bagel to spare, this is a fun exercise to try before reading further. Colin Adams in *The Knot Book* (a wonderful resource for those interested in learning more about mathematical knot theory) gives a method for drawing a (P,Q) torus knot that involves marking and connecting equidistant points along the inner and outer longitudinal “equators” of the torus. In bead crochet, we are “drawing” with beads and are thus constrained by bead pixelation, so we can’t draw smooth curves or lines anywhere we want. Despite these added constraints, working in bead crochet we have a wonderful advantage over someone trying to draw lines on, say, a hard ceramic donut. Crocheted rope has flexibility and stretch to it, a feature that topologists dream about, and one that we will use to great advantage in our first set of constructions. As observed in Chapter 1, topologists study geometric properties that don’t change with stretching and shrinking†— so the classic joke about them is that they don’t know the difference between a donut and coffee cup … because either one can be deformed by stretching and shrinking into the other!

Overall, our approach is based upon either physical twists of the rope, natural spiraling of the pattern, or a combination of both. Before discussing bracelet design methods in detail, let’s forget about bead crochet briefly and get oriented conceptually with a related pencil and paper “thought experiment” on how to construct a torus knot starting from a flat torus. This is only a thought experiment because our paper needs to be flexible and stretchy while still holding its shape, and most likely you don’t have any such paper on hand (and neither do the topologists, except in their imaginations). After our experiment, we’ll see how this method applies to bead crochet.

Consider a square flat torus with Q evenly spaced parallel vertical lines drawn on it, as shown upper left in Figure 4.5 for Q = 3. Now imagine gluing the identified edges together: first the right and left edges of the paper to produce a cylinder and then the top and bottom edges to produce the torus (using the method depicted in Figure 1.2). What happens to the vertical lines? The top row of Figure 4.5 illustrates this process. We can see that the lines on the flat paper form three separate, evenly spaced, longitudinal rings on the torus. You may notice this is related to the geometric cross-section designs described in Chapter 3. Now suppose we try it again, but give the cylinder a 1/3 twist before connecting the ends. What happens to the “vertical” lines now? Marvelously, we have connected the first line to the second, the second to the third, and the third back to the beginning again—to form a single loop that wraps around the torus three times longitudinally while at the same time wrapping just once meridionally, all without ever crossing itself—in other words, we have constructed a (1,3) torus knot, as shown in the bottom row of Figure 4.5!

This same method works for any values of relatively prime P and Q; a P/Q twist before closing on a cylinder with Q vertical lines will always produce a (P,Q) torus knot!


† As opposed to, for example, tearing or puncturing.

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However, you may need some further convincing to see why this is true and to gain insight into what happens when P and Q are not relatively prime. For this, we’ll turn to star polygons, adapting a lovely instructional idea we learned about from Sandy Spitzer at the 2012 Bridges Conference (a yearly interdisciplinary conference on mathematics and the arts)¹ Sandy presented a paper on using star polygons to provide students with intuition for understanding cyclic groups in mathematics. We noticed that they also provide a nice model for conceptualizing torus knot construction in bead crochet.

We construct a \((P/Q)\) star polygon on a circle with Q equally spaced perimeter points by drawing lines connecting every Pth point. The construction process starts with any one of the perimeter points, and a link is drawn between it and the point P clockwise dots away. The next link is drawn from that point to the following point P clockwise dots away, and so on, until we return to the starting point. When P and Q are relatively prime, this process is guaranteed to include all Q perimeter points, and the resulting figure is called a regular star polygon.² Figures 4.6 and 4.7 show all possible star polygons produced on a circle with 8 points and 7 points, respectively. Note that different values of P sometimes produce the same star polygon.

Testing star polygon construction using a variety of different values for P and Q should give you some valuable intuition for how the process behaves differently depending on the values of P and Q and, in particular, why relatively prime P and Q are needed to include all perimeter points.

For modeling purposes, we will think of the star polygon circle as the place in our torus knot construction where the two cylinder ends meet. The choice of Q represents the number of vertical lines on the cylinder, and the circle is a cross section (or overhead view) where all you can see is the circular cylinder end and dots at the endpoints of the vertical lines. The choice of P represents the amount of twisting prior to closing the two ends of the cylinder. Since the dots

**FIGURE 4.5** Thought experiment: starting with a flat torus with three evenly spaced vertical parallel lines, connect the left and right edges to roll it into a cylinder. Next, connect the top and bottom edges (now the circular ends of the cylinder) to form a torus. What happens to the three lines? Now try it again, but give the cylinder a 1/3 twist before closing. What happens to the lines now?

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² If P and Q are not relatively prime, this process will not include all the perimeter points, in which case we obtain a star polygon for a smaller value of Q. For example, in Figure 4.6, the (2/8) star polygon is also a (1/4) star polygon. If the drawing process is begun again at one of the unused perimeter points, and this is repeated until all perimeter points have been used, the result is called a star figure (see http://mathworld.wolfram.com/StarFigure.html). Star figures have a connection to another construct from topology called a torus link. We do not discuss torus links in this book, but they are also interesting and can be represented in bead crochet using the techniques described here.
divide the circle into $Q$ evenly spaced sections, a connection between a given point and the point $P$ dots clockwise away from it models a clockwise $P/Q$ twist of the cylinder end. Thus, lines drawn between points show how the cylinder's vertical lines connect with one another at a closure sewn with a $P/Q$ twist. For example, as shown middle right in Figure 4.6, a circle with eight evenly spaced perimeter points ($Q = 8$) linking every third point ($P = 3$) models the connections on a cylinder with eight vertical lines that is closed with a $3/8$ twist at one end. Since for these values of $P$ and $Q$, all eight of the perimeter points are included in the star, all eight of the original vertical lines would be connected in a single knot.

How many meridian traversals does a $3/8$ twist produce? The physical twist of the rope causes each of the eight vertical lines to travel $3/8$ of the way around the meridian. Summing all those partial traversals, we can calculate that the entire knot will travel around the meridian a total of $8 \times 3/8 = 24/8$ times for a total of three full meridian traversals. Thus the eight vertical lines of our cylinder will connect into a single knot that simultaneously travels around the torus eight times longitudinally and three times meridionally, which is precisely the definition of a $(3,8)$ torus knot!

An alternate way of using a star polygon model to calculate the number of meridian traversals is to count how many times your hand has to travel around the circle during the drawing process. This also helps distinguish between identical star polygons with differing values for $P$. For example, during the drawing process of a $(3,8)$ star polygon, as you count out dots to skip while moving clockwise in sequence, your pencil will travel exactly three times around the circle before returning to the starting point again. By contrast, if you try constructing the $(5/8)$ star polygon, modeling a $5/8$ twist, you will get the same regular star polygon, but this time your pencil will travel five times clockwise around the circle. So it is not just the final form of the star polygon that is informative, but also the process of creating it that helps model and visualize what is happening as we introduce different twists before closing the cylinder ends of the rolled flat torus.

What happens if $P$ and $Q$ have a common divisor, or in other words, are not relatively prime? For example, let’s try $P = 2$, which divides evenly into 8 and represents a $2/8$ (or $1/4$) twist. In this case we get the star polygon shown middle left in Figure 4.6, in which only half the points on the circle wind up interconnected. What kind of knot is constructed now? By playing with star polygons you should be able to make several observations about their behavior. The first and most important is that only when $P$ and $Q$ are relatively prime are all $Q$ points in the circle included in the polygon. Hence only in this case is a single torus knot produced that uses all $Q$ vertical lines.

From a star polygon perspective, a clockwise twist of $P/Q$ is functionally equivalent to a counterclockwise twist of $1 - (P/Q)$, e.g., $1/8$ clockwise and $7/8$ counterclockwise produce the same star polygon. In fact, so do all twists of $1/8$ and $7/8$, regardless of direction. Similarly, twists of $1/8$, $9/8$, $17/8$, etc., all produce the same star polygon. This does not mean that the torus knots produced with these twists are identical, only that the final star polygons are. The torus knots are all identifiably different, depending on the direction of twist and number of meridian traversals.

Now that we know how to construct a specific $(P,Q)$ torus knot with our hypothetical stretchy paper by drawing...
vertical lines and applying the appropriate $P/Q$ twist before closing the cylinder, we can use this idea to create torus knots in bead crochet. As mentioned earlier, our thought experiment suggests a method based on the geometric cross-section designs in Chapter 3. Using these designs, for odd values of $Q$ we can construct perfectly vertical dotted lines equally spaced on bead crochet rope. Using drop beads for the spines can give the lines a more solid appearance. For example, the Equilateral Triangle 7-around design shown on p. 41 creates $Q = 3$ vertical lines, and the Pentagon design on p. 42 creates $Q = 5$ vertical lines.

But can we achieve a twist in the rope as desired before closing to create a torus knot? For the types of twists we want, the answer is yes. A single full twist, although insufficient by itself for our purposes, is generally no problem for a bracelet length rope—in fact, if you have made a lot of bead crochet ropes, you’ve probably inadvertently closed bracelets with a full twist already. Multiple full twists can tax the flexibility of the rope and might be more easily noticed, but if you do not limit yourself to bracelets—if long necklaces will do—you can generally achieve multiple full twists without reaching the physical torque limit of bead crochet.

However, by themselves, single or multiple full twists are not useful because they always connect each vertical line directly back to itself, so the ratio represented by the twist can never have relatively prime $P$ and $Q$. For example, two full twists with a $Q$ of 3 would give a ratio of 6/3, and 6 and 3 are not relatively prime. Thus, for our purposes, we also need smaller partial twists like 1/3, 2/3, or 3/5. How can we accomplish those? Fortunately, the regular geometric cross-section designs are constructed symmetrically so that each repeat is composed of $Q$ smaller identical sections, each effectively adding $1/Q$th of the rope’s circumference. By using a single color for all sides of the polygon instead of a different color for each side, we can make these smaller sections of the repeat identical to each other. This leaves us with a single background color and a contrasting spine color that represents the “string” for our knot. Then, due to the symmetry of the cross-section designs, it is no problem to leave off (or add on) one or several of the small identical sections, and doing so will induce various fractional twists with $Q$ in the denominator. If we need a twist greater than one, we can accomplish this by combining full physical twists of the rope with these smaller twists induced by partial repeats. For example, a 5/3 twist is the same as a 1 and 2/3 twist, and it can be accomplished by combining a full twist plus a 2/3 twist induced by omitting 2/3 of the final repeat.

**FIGURE 4.8** A (5,3) torus knot bracelet produced from the equilateral triangle cross-section design on p. 41, using drop beads and size 11 seed beads in 7-around. The final repeat on the bracelet uses only 7 of the 21 beads in a repeat to induce a partial twist of 2/3. This is added to a full twist to produce the requisite 5/3 twist before closing.

The bracelet shown in Figure 4.8 is an example of exactly this approach based on the equilateral triangle 7-around design. In this case, the 21-bead repeat of the original triangle pattern is portioned into three identical 7-bead sections, and the final repeat is reduced by two sections to only 7 beads, thereby enabling a 2/3 counterclockwise twist. This 2/3 twist was added to a full physical counterclockwise twist to produce the 5/3 twist needed to get a (5,3) torus knot.°

Using partial repeats to produce fractional twists and possibly one or more full physical twists, we can achieve a $P/Q$ twist of a geometric cross-section design with $Q$ faces. Keep in mind, however, that, depending on stitch gauge, circumference, and bead type, multiple full twists can create too much torque in a bracelet-sized rope, so a longer necklace-length piece could be needed.

There is nothing like playing with the bracelets themselves to clarify these ideas. If the math seems confusing, try experimenting with different twists and partial repeats on one of the regular cross-section designs. **Figure 4.9** shows the 4-around Equilateral Triangle bracelet (pattern on p. 138) and all the $(P,3)$ torus knots that it can produce up to $P = 7$. We made all the bracelets shown from the same crocheted piece, twisted, closed, reopened, retwisted, and

° Note that reducing the last repeat to only 7 beads could alternatively have been used to induce a 1/3 twist clockwise, so the same exact bracelet could also be closed into a $(1,3)$, $(2,3)$, or $(4,3)$ torus knot, depending on the direction chosen for the induced twist.
closed again to try out all the various twist and partial repeat options (which is why the thread tail is visible in the photos). This particular bracelet ends with only two-thirds of the final repeat (using 6 of 9 beads in the last repeat), forcing either an N + 1/3 twist in the counterclockwise direction or an N + 2/3 twist in the clockwise direction. A 1/3 twist produces a (1,3) knot, a 2/3 twist a (2,3) knot, a 4/3 twist a (4,3) knot, etc.

So far, our examples have used only the odd geometric cross sections (triangle, pentagon, etc.) because those are the designs that produce perfectly vertical lines. However, we can also use the even cross sections (digon, square, etc.), in which the pattern has a natural slant to it. The trick is first to straighten out the natural slant of the spines by untwisting them (as we might normally do for these designs), and only then to apply the desired P/Q twist. Figure 4.10 shows two (3,4) torus knot bracelets created using this approach, based on the Square 6-around pattern on p. 142. With this trick in mind, it should also be possible, for example, to figure out how to achieve a (3,2) torus knot using as a basis the Möbius band designs in Chapter 3. In fact, the Möbius band bracelet itself is already a (1,2) torus knot design!

Although both bracelets in Figure 4.10 are (3,4) torus knots, you may notice that they have an interesting difference: their knots spiral in opposite directions, one clockwise, the other counterclockwise. In fact, they are mirror images of one another. You may also have noticed that the spirals in the Figure 4.9 set of bracelets alternate direction as P increases. These observations are related to another tidbit from knot theory, namely, that there are two distinct variants of a (P,Q) knot: a knot and its mirror image.*

The exception to this rule is when P or Q is equal to one, in which case the mirror versions are actually the same knot, an unknotted loop.

* The exception to this rule is when P or Q is equal to one, in which case the mirror versions are actually the same knot, an unknotted loop.
The mirror image bracelets in Figure 4.10 are both (3,4) torus knots done in 6-around with drop beads (in white) and size 11 seed beads (in brown), based on the Square 6-around design on p. 45. You can think of them as two related species of the same torus knot.

When we first considered the problem of constructing mirror image but otherwise identical torus knots in bead crochet, we originally believed the only possible construction method was to use the same pattern and crochet one left handed and the other right handed, which forces the spirals to run in opposite directions. If you are ambidextrous, or can team up with an opposite-handed partner, right- and left-handed crocheting is perhaps the ideal solution to this problem. Teaching yourself to crochet both ways has the added benefit of enabling you to teach both left- and right-handed friends to bead crochet. Eventually, however, we realized a less taxing solution was possible, as demonstrated by the bracelet pair in Figure 4.10, which avoided resorting to left- and right-handed crocheting. As an extra credit challenge, try figuring out exactly how this mirror image pair was constructed. A hint is that one bracelet has 10 more beads in it than the other (and both end with fewer than the entire 20-bead repeat in the pattern).

If you are picky, you might therefore complain that the two bracelets are not true identical mirror images. However, the casual observer would never know, so we’re quite pleased with them. We’ll have a chance to play a bit more with mirror image knots in Chapter 5 on knotted and linked bracelets.

A final interesting side note on star polygons is that it is also informative to construct them with P and Q reversed in meaning. This figure shows a sequence of steps relating a (5/3) star polygon—constructed with five perimeter points on a circle with links connecting every third point—to a (5,3) torus knot. The image on the right was generated by the Knotplot software created by Rob Scharein.

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Construction Using Natural Twists

So far so good—we’ve proposed a successful approach for solving the torus knot challenge in this chapter. However, these designs rely heavily on physical twisting, which has some annoying limitations in bead crochet. Is there another construction method that doesn’t depend so much on physical twists, but instead develops a pattern that spirals naturally at just the right slope and can thus be closed without any significant torquing of the rope? One benefit of untwisted bracelets is full preservation of the rope’s natural flexibility, which is one of bead crochet’s most appealing tactile qualities.

How can we understand and control the natural spiral of a pattern to create torus knot designs that wrap longitudinally and meridionally according to a specific desired (P,Q) plan? This is exactly the question we address next, using a different construction method that eliminates the need for any significant physical twisting. Here again, before describing how to do it in bead crochet, we return to a flat torus thought experiment to see how it might be done in principle. The approach might take a little mind-bending conceptually, but it makes good intuitive sense once you understand the ideas.

For this thought experiment, imagine constructing a patchwork of flat tori that is P tori wide and Q tori high, as shown on the left in Figure 4.12 for P = 3 and Q = 2.
The result is a (3,2) torus knot drawn on the torus surface. On the right, Figure 4.12 shows the transparent, infinitely thin, stacked tori resulting from this process. Figure 4.13 shows, bagel-rendered, the final result of our imaginary process applied to the patchwork in Figure 4.12.

So, how do we extend this idea from thought experiments and bagels to a practical torus knot construction in bead crochet? Figure 4.14 illustrates a construction for a (4,3) torus knot design using vertical bracelet layouts as the patchwork elements. To simplify things, we’ve used an untwisted bracelet (i.e., a vertical layout with an integral number of double rows); but as we’ll see later, this is not required. In this example, the layout length is short, only 8 double rows, which is clearly not long enough (a bracelet made with size 11 seed beads typically requires at least 50 double rows), but it suffices to demonstrate the idea.

A decision is clearly required about which beads to “paint” to represent the straight line. Given the coarse pixellation of bead crochet, which precludes painting partial beads, the line typically needs to zigzag to stay on course. For symmetry and aesthetics, we want a short zigzag that closely tracks the slope of the diagonal and a rope length that provides space for an integral number of these zigzags, as shown in the example in Figure 4.14.

Although the design in Figure 4.14 is not long enough for an actual bracelet, we can use it anyway, repeating it multiple times to get a longer rope. To get a good size for a bracelet, 52 double rows work nicely. Since the original patchwork design creates a traversal of the meridian every 2 double rows, 52 double rows result in 26 meridian traversals or a (26,3) torus knot. In this particular bracelet, however, we chose to play with the twist as well by untwisting some of those traversals to generate the interesting looking (20,3) torus knot shown at the bottom of Figure 4.15.

\[ \text{For additional insight into the importance of having relatively prime } P \text{ and } Q, \text{ try drawing some patchworks with diagonal lines using } P \text{ and } Q \text{ that are not relatively prime, such as } P = 6 \text{ and } Q = 4, \text{ and see how those patchworks behave.} \]

\[ \text{Recall from Chapter 1 that a vertical layout is composed of alternating rows of } N + 1 \text{ and } N \text{ beads, where } N \text{ is the rope circumference. A double row is defined as a chunk of two consecutive rows in a vertical layout, i.e., a row of } N + 1 \text{ and a row of } N \text{ beads.} \]
FIGURE 4.13  Left: A bagel rendition of the (3, 2) torus knot from the patchwork in Figure 4.12. The blue portion of the line is there, too, but can be seen on the underside only. Right: a modified “reversed” star polygon diagram (similar to those in Figure 4.11) of the same knot with the underside portions shown as dotted lines.

FIGURE 4.14  A (4, 3) torus knot construction using a four by three patchwork of “full” bracelets in vertical layout form. The bracelet of 8 double rows here suffices for an example of the technique, but is too short for a wearable bracelet. For the construction, imagine the patchwork material is infinitely thin and transparent. To connect the corresponding interior points, patchwork components are stacked up like a deck of transparent cards to display the complete pattern in a single vertical layout, shown at right.
Figure 4.16 shows the patchwork method applied to a (3,2) torus knot design on a full bracelet-sized patchwork. This design is also provided in the pattern pages as the Zigzag (3,2) Torus Knot on p. 156. Although our choice of zigzag here resulted in the “line” stepping out of the patchwork bounds at the top end, this is not a problem, because there really are no patchwork bounds. If you visualize an additional patchwork element on the upper right you can see where that snippet of the zigzag belongs on the full single layout. As long as the line begins at the lower left corner of the original patchwork and ends at the upper right corner (which is another way of saying it must have a slope of $Q/P$ on the bead crochet grid), and the zigzagging of the line does not cause it to intersect itself, the design will work. Too narrow a bracelet circumference or too wide a zigzag could cause problems with self-intersection that would not arise with a straight, thin line, but in this case, the design works quite well.

Figure 4.15 A full-size bracelet made using the construction and design shown in Figure 4.14. With 52 double rows and a twist, the design produced a (20,3) torus knot (at the bottom). This pattern, Long Zag (P;3) Torus Knot, appears on p. 158.

Figure 4.16 The patchwork and final bracelet design for the Zigzag (3,2) Torus Knot pattern on p. 156 and shown bottom on the pear of the chapter header photo, p. 50.
We have glossed over several technical difficulties that can arise with this type of construction method. One is that creating a large full-sized patchwork like the one in Figure 4.16 can be unwieldy, possibly requiring poster-sized bead crochet graph paper. A second difficulty is that to achieve a symmetric design we likely need to tinker with the bracelet circumference and length parameters to find a patchwork whose diagonal has room for an integral number of the small repeating zigzag elements. Unfortunately, the unwieldiness of patchwork construction makes this type of trial and error unappealing. While the patchwork idea is useful for conceptual understanding, it is possible to determine the slope and length of a patchwork diagonal line using simple numerical calculations and thereby avoid patchwork construction altogether. Once we have chosen a repeating zigzag design that tracks a particular slope, its exit point on one side of a vertical layout determines (via a hockey-stick translation) where it enters again on the opposite side. So the patchwork is not needed for this task either since we have an alternate method to determine the complete course of a specific zigzag pattern on a full bracelet layout.

On a square flat torus, determining the slope of the diagonal is straightforward because the slope needed for a \((P,Q)\) torus knot is simply \(Q/P\) (i.e., a rise of \(Q\) over a run of \(P\)). Unfortunately, a bracelet vertical layout is not square and, complicating matters further, the relevant units on the horizontal and vertical axes are different. So our slope calculation is a bit more complex. For readers interested in trying this “slope calculation approach” who don’t mind a few gory technical details, the remainder of this section describes a method of calculating both the length and slope of the diagonal based on the patchwork input values of \(P, Q\), the bracelet length \(L\), and the circumference \(N\).

Our first task is to define a coordinate system on a bead plane grid that measures columns of beads on the horizontal axis and rows of beads on the vertical axis. Figure 4.17 shows such a coordinate system. Note that, in bead crochet, the columns are separated by half the width of a bead and the rows by the height of a bead, which accounts for the different units on the two axes. The adjacent vertical layout elements in a patchwork are placed a hockey-stick translation away from one another, as marked by the blue and green beads in Figure 4.17. For bracelets of circumference \(N\) on this coordinate system, the hockey-stick translation adds \(2N + 1\) in the horizontal direction and subtracts 1 in the vertical direction.

To construct a \((P,Q)\) knot, consider a \(P\) by \(Q\) patchwork where

\[
L = \text{bracelet length in double rows, and} \\
N = \text{bracelet circumference.}
\]

Using the bead plane coordinate system defined in Figure 4.17, each of the \(Q\) layout elements in a patchwork adds \(2L\) in the vertical direction. However, due to the hockey-stick translation, the Y-coordinate of the upper right point on a patchwork diagonal is shortened by one for each of the \(P\) elements in the horizontal direction. Thus the patchwork rise is \((2LQ) - P\). Each of the \(P\) elements in the patchwork adds \(2N + 1\) in the horizontal direction. Thus the patchwork run is \(P(2N + 1)\). So the slope of the
diagonal line running from the lower left to the upper right corners of a patchwork (i.e., rise/run) is equal to

\[
\frac{(2\ell Q) - P}{P(2N + 1)}
\]

For a symmetric zigzag that closely tracks this slope, we need a rise and run with a common divisor such that rise/run is reducible to a ratio with relatively small whole numbers in the numerator and denominator. We can attempt to find such a ratio by adjusting our choices for the values of \(N\) and \(L\). Tinkering with the length of the full bracelet, \(L\), is feasible, since there is always some limited flexibility in this parameter. Tinkering with the circumference \(N\) is also an option, and in this regard, it’s useful to note that \(2N + 1\) is prime for 5-, 6-, 8-, 9-, and 11-arounds and factorable for 7-, 10-, and 12-arounds. So these latter circumferences are more likely to produce a reducible slope.

Let’s try the method on an example with \(P = 4\) and \(Q = 3\). If we choose \(L = 49\) and \(N = 10\), we get \(2\ell Q - P = (2)(49)(3) - 4 = 290\) and \(P(2N + 1) = 4(21) = 84\). So rise/run = \(290/84 = 145/42\), which doesn’t reduce any further. However, if we increase \(L\) to 52, we suddenly get a rise/run of 308/84 = 11/3, which has much smaller numbers in the numerator and denominator. This permits zigzag segments separated by 11 beads vertically and 3 half beads horizontally that fit perfectly in the 52 double rows of a full bracelet. Using a single full 52-double-row vertical layout as our canvas, we can place these grid beads (the yellow beads in Figure 4.18) accordingly. Once the grid is laid out, we can experiment with different zigzag options on it, taking care to choose one that does not overlap itself. Four possibilities are shown in gray in the upper right. We can then experiment further with widening or otherwise modifying the design as shown in the progression in Figure 4.18. Figure 4.19 shows two resulting bracelets, the expected (4,3) torus knot done on the planned 52 double rows in size 11 seed beads and a (5,3) torus knot that resulted from using smaller size 11 Delica beads on a longer rope (i.e., with a larger \(L\)). These bracelets, which have an identical foreground and background pattern around the accent beads, also offer a taste of the Escher designs coming up in Chapter 6.

### Combining Patchworks and Twists

All the patchwork examples above use “untwisted” bracelet components (i.e., using the terminology of Chapter 1, they are structurally aligned). However, as shown in Figure 4.20, this is not necessary. On the left is an example with untwisted components. On the right is an example in which the bracelet components are three beads short of a full double row at the top, forcing a slight twist at the bracelet closure. In this latter case, we can still create a patchwork diagonal, and the small twist at the closure makes up for the lost bit of span in this diagonal. In this particular example, the diagonal happens now to line up perfectly with a natural diagonal of the bead plane, so no zigzagging is even needed (although this won’t be true in most cases). One drawback to using twisted patchwork components is that it can further complicate the slope calculations described above.

In general, it’s quite useful to combine both physical twists and natural twists to achieve a desired knot. For example, consider the Zigzag (3,2) Torus Knot on p. 61. For sizing purposes, you might want to adjust the length of the bracelet to be a bit shorter by using fewer than 45 repeats. Unfortunately, doing so removes a portion of the final meridian traversal. However, strategically leaving off a few repeats permits a tiny physical twist to replace the missing bit of natural pattern twist. In this case, as long as the total number of repeats is odd, you can omit a few repeats and still obtain the desired (3,2) knot with the correct choice of twist. Since physical twists run into problems with torquing the rope and natural twists run into sizing constraints, combining both techniques is often the best approach to get exactly the knot you want.

Sometimes, even armed with the slope calculations from the previous section, it becomes clear that a nicely reducible slope is simply not achievable on a bead crochet grid for a workable rope length and circumference. In this case, the only practical solution is to choose a slope that is as close as possible to the required one, and then rely more heavily on twisting to achieve the desired knot.

A final point worth mentioning is that we can use the natural slant of many existing patterns to produce torus knots designs. However, we might need a reverse engineering of the slope calculations described above to figure out in advance exactly what torus knot(s) would result. Alternatively, we can avoid the math and try simply making the bracelet to find out! Consider, for example, the hexagonal pattern shown in Figure 4.21, which you may recognize from its frequent appearances in Chapter 1. It turns out that it generates the Zigzag (3,2) Torus Knot trefoil design on p. 61, as shown in the bottom example in Figure 4.21.

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* Reverse engineering would start with the known slope, \(L\), and \(N\) and calculate \(P\) and \(Q\) from them.
CHAPTER 4

FIGURE 4.18 (4,3) Torus Knot design and transformation. On the left is one possible complete bracelet design for $P = 4$, $Q = 3$, $L = 52$, $N = 10$ with a calculated slope of $11/3$ on the bead crochet coordinate system. Note that this slope is easily seen in the yellow accent beads, which are separated from one another by 11 beads vertically and 3 half beads horizontally. The basic design can then be transformed as desired while still maintaining this same slope, as shown on the right. In this case, it has been transformed into two identical interlocking (4,3) torus knots in each color with the yellow accent beads in between.
FIGURE 4.19 (5,3) (left) and (4,3) (right) torus knot bracelets created from the design developed in Figure 4.18. The pattern, Knotted Snakes, appears on p. 159.

FIGURE 4.20 It is possible to create a patchwork diagonal even when the patchwork elements are not composed of an integral number of double rows. In this case, a small amount of physical twist makes up for the smaller run of the diagonal line.

FIGURE 4.21 Generating torus knot patterns from the natural slant of existing tilings/patterns.
However, we can use the same pattern to generate other (3,2) torus knot designs with a different zigzag, such as the one in the top example.

You should now have plenty of tricks up your sleeve for producing almost any sort of bead crochet torus knot you desire! Using the techniques we've described, Figure 4.22 circles back to where we began. Recreating in bead crochet a variation of Figure 4.3 from the start of the chapter, it displays a window into the same conceptual infinite graph, although this time the axes are swapped, with P on the vertical axis and Q on the horizontal axis. For P between 3 and 7 and Q between 2 and 4, wherever the (P,Q) pair is relatively prime, it shows a bead crochet (P,Q) torus knot bracelet and also a hand-drawn, "reversed-meaning," (P/Q) star polygon. In the remaining spots, where the (P,Q) pair is not relatively prime and thus neither a (P,Q) torus knot nor (P/Q) star polygon is possible, it shows a compound star figure with component star polygons in separate colors.

Titled The Torus Traveler’s Journey, this wall art piece is intended to invite the viewer to ponder connections between torus knots and star polygons.

FIGURE 4.22 The Torus Traveler’s Journey, an art piece included in the Joint Mathematics Meetings Exhibition of Mathematical Art in January 2014. All but two of the patterns used are provided in our pattern pages: the (3,2) on p. 156, the (5,2) on p. 161, the (4,3) and (5,3) on p. 159, and the (P,4) knots on p. 162.

* In addition to swapping axes, it displays a smaller window than the one in Figure 4.3. Figure 4.22 can be mapped onto Figure 4.3 by rotating it 90° counterclockwise and then flipping the Q axis so the smaller numbers are at the top.